SPRING 2025 MATH 540: QUIZ 12

Name:

1. Show that if a and b are positive integers, then the arithmetic progression $\{a, a+b, a+2b, \ldots\}$ contains an arbitrary number of consecutive terms that are positive.(5 points)

Solution. Note that the general term of the sequence is $a + bn$, and no term in the sequence is equal to 1. Fix $k \geq 1$ and take $n_1, n_2, \ldots, n_k$, k consecutive terms in the sequence. We claim that for any $t \geq 1$, $tb + n_1, tb + n_2, , \ldots, tb + n_k$ are also consecutive terms in the sequence. Suppose the claim is true. Set $t := n_1 \cdots n_k$. Then $tb + n_1$ is divisible by n_1, $tb + n_2$ is divisible by n_2, $\ldots, tb + n_k$ is divisible by n_k. Note that for each i, $tb + n_i = n_i(\frac{t}{n_i}b + 1)$, showing that $tb + n_i$ is not prime. So for arbitrary k, we have k consecutive composite terms in the sequence.

For the claim, suppose $n_1 = a + hb, n_2 = a + (h+1)b, \ldots, n_k = a + (h+k-1)b$. Then,

$$\begin{aligned}
tb + n_1 &= tb + (a + hb) = a + (h+t)b \\
tb + n_2 &= tb + (a + (h+1)b) = a + (t+h+1)b \\
&\vdots \\
tb + n_k &= tb + (a + (h+k-1)b) = a + (t+h+k-1)b,
\end{aligned}$$

which are k consecutive terms in the sequence.

2. Calculate $\Phi_{24}(x)$. (5 points)

Solution. We use the fact that $x^{24} - 1 = \Phi_1(x) \cdot \Phi_2(x) \cdot \Phi_3(x) \cdot \Phi_4(x) \cdot \Phi_6(x) \cdot \Phi_8(x) \cdot \Phi_{12}(x) \cdot \Phi_{24}(x)$. From class we have $\Phi_1(x) = x - 1, \Phi_2(x) = x + 1, \Phi_3(x) = x^2 + x + 1, \Phi_4(x) = x^2 + 1, \Phi_6(x) = x^2 - x + 1$. We use this to find $\Phi_{12}(x)$. We have

$$x^{12} - 1 = (x-1)(x+1)(x^2+x+1)(x^2+1)(x^2-x+1)\Phi_{12}(x)$$

Solving for $\Phi_{12}(x)$, we get $\Phi_{12}(x) = x^4 - x^2 + 1$. We now have,

$$x^{24} - 1 = (x-1)(x+1)(x^2+x+1)(x^2+1)(x^2-x+1)(x^4-x^2+1)\Phi_{24}(x).$$

Solving for $\Phi_{24}(x)$, we get $\Phi_{12}(x) = x^8 - x^4 + 1$.